On the melting of a semi-infinite body of ice placed in a hot stream of air

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SUMMARY

A simple mathematical model is proposed to describe the steady melting of a body of ice which presents a plane surface transverse to a stream of hot air; the temperature of the air is such that vaporization does not occur.

The analysis takes into account the convection of heat away from the surface by the water released in melting and the results show that the rate of transfer of heat to the body and thus the rate of melting, is reduced by as much as 46% by this convection.

Simple approximate expressions are obtained for the rate of melting, the thickness of the water layer, and the thickness of the thermal boundary layer in the ice, in terms of a basic parameter S which can be calculated in terms of known quantities. These results are compared with those obtained by a separate Pohlhausen calculation and are found to be in good agreement.

It is also shown that there exists a thermal boundary layer, in the body, of thickness much greater than that of the boundary layer in the air, in which the temperature changes rapidly from its value at the melting surface to its value in the far interior.

1. INTRODUCTION

When a non-insulated body is placed in a stream of hot air a transfer of heat to the body takes place; when the temperature of the stream is high enough the body may melt and even vaporize. The rate at which the body melts is determined by the amount of heat available for latent heat; this is the difference between the total amount of heat transferred to the surface and the amount conducted away from the surface to the interior of the body.

The present paper considers the steady melting of a semi-infinite body of ice whose plane surface is placed normal to a stream of hot air (see figure 1); the analysis may be regarded as applicable to the conditions in the neighbourhood of the forward stagnation point of a blunt-nosed body of arbitrary shape. The temperature range considered is such that vaporization does not occur. It is possible, under certain simplifying assumptions, to reduce the Navier–Stokes and energy equations to two ordinary differential equations which are solved by an approximate method for both the plane and axisymmetric flows.

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An interesting and important feature of the flow is the shielding effect of the thin layer of water which forms between the ice and the air. The rate of heat transfer to the body, and thus the rate of melting, is reduced in two ways by the presence of the water layer: firstly, the temperature of the air–water interface is raised above that of the melting surface, which tends to reduce the rate of heat transfer from the air, and secondly, convection of heat takes place in the water layer which further reduces the rate of heat transfer across the melting surface.

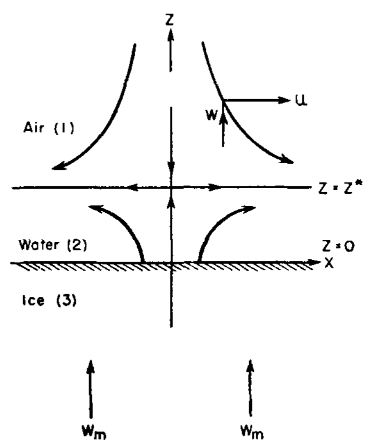

Figure 1.

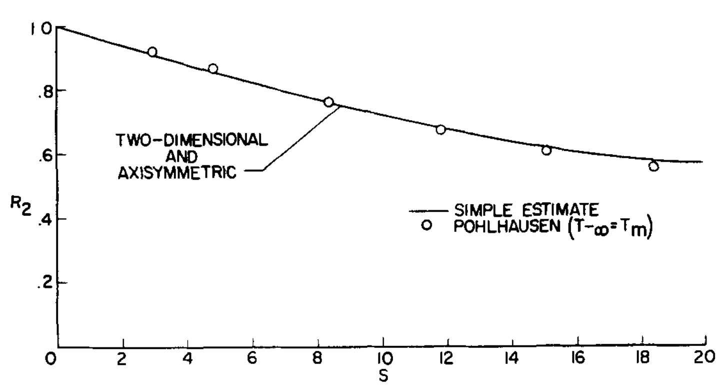

Figure 2. Graph of the shielding ratio R_2.

It is found that the first shielding effect is small, but that the second shielding effect may become appreciable. In figure 2, there is shown the shielding ratio R_2, defined as

$$R_2 = \frac{\text{rate of transfer of heat from water to ice}}{\text{rate of transfer of heat from air to water}},$$

vs the basic parameter S defined by

$$S = \frac{c_{p1}(T_\infty - T_m)}{L + c_3(T_m - T_{-\infty})}$$

where c_{p1} is the specific heat of air, c_3 is the specific heat of ice, L is the heat of fusion of ice, and T_∞, T_m and $T_{-\infty}$ are respectively the temperature of air at large distances from the body, the temperature of water at the melting surface (taken always to be 0° C), and the temperature in the far interior of the body. (The temperature T^* at the air–water interface also plays an important role in the theory, but has been eliminated from the final results.)

It should be noticed that the shielding effect is largely independent of the flow conditions, since for ordinary conditions, the shear acting on the water layer is always sufficiently large to carry the water away at a reasonable speed compared with the rate of melting. On the other hand, the details of the flow in the water layer, such as its thickness and the rate at which the ice melts, do depend on the conditions of flow.

2. Description of the flow and method of solution

2.1. *The flow configuration*

The flow considered is that shown in figure 1. In order to reduce the problem to a steady state the coordinate system chosen has axes fixed in the melting surface; in this coordinate system the interior of the ice moves towards the stationary melting surface, $z = 0$, with constant velocity, w_m, equal to the rate of melting.

Considerations of the continuity of mass lead us to expect (under the assumptions of § 2.2) a water layer of constant thickness between the ice and the air stream. From the Hiemenz solution for flow near a stagnation point (which will be recalled in detail later) is is known that the horizontal component of velocity in the air is proportional to the horizontal distance x from the stagnation point, and it follows that this is also true of the motion of the water since the velocity is continuous at the interface. Also, since the rate of heat transfer from the air, in the vertical direction, is independent of x, it is to be expected that the body will melt at a rate which is independent of x. Thus the amount of water crossing a vertical line at the position $x = x_1$ is that which has been produced by melting between the line of symmetry $x = 0$ and the line $x = x_1$; this amount is proportional to x_1. The ratio of the amount of water to the velocity is constant, i.e. the thickness of the water layer is constant.

Because of the continuity of velocity and stress at the air–water interface $z = z^*$, the motion of the air is affected only slightly by the presence of the water. (It will be shown that the velocity of the air at the interface is of the order of the square root of the density ratio, $(\rho_{\text{air}}/\rho_{\text{water}})^{1/2}$ which is $O(10^{-2})$ for most of the temperature range under consideration.) The Hiemenz solution, modified to take account of the variation with temperature of the density, viscosity, and thermal diffusivity of air, is a good approximation for the flow now under consideration. In a region near the stagnation point the flow velocities are small and compressibility effects are ignored.

In the water it is assumed that the density and the thermal diffusivity are constant but the viscosity varies with temperature.

2.2. *The basic assumptions*

Only the case of laminar flow is considered in this paper. In addition, the following approximations are made in order to simplify the analysis.

(1) The magnitude of the components of velocity in the air are such that the approximation $M = 0$ (M = Mach number) may be made. The equation of state is then $\rho_1 T$ = constant, where ρ_1 is the density and T is the absolute temperature.

(2) The viscosity μ_1 and thermal conductivity k_1 of the air obey the laws

$$\frac{\mu_1}{T} = \text{constant}, \qquad \frac{k_1}{T} = \text{constant}.$$

(3) The specific heat of air c_{p1} is constant.

(4) The Prandtl number for air, σ_1, is constant (this is implied by (2) and (3)).

(5) Viscous dissipation is negligible.

(6) The density ρ_2, thermal conductivity k_2, and the specific heat c_2, of water are constant, but the viscosity μ_2 varies with temperature.

(7) The melting temperature T_m, the specific heat c_3, the thermal conductivity k_3, and the density ρ_3, of ice are all constant.

2.3. *The method of solution*

The solution is developed in the three regions, $z > z^*$ (air), $z < 0$ (ice) and $0 < z < z^*$ (water). At the interfaces $z = z^*$ and $z = 0$ certain quantities which appear in the boundary conditions are not known, *a priori*, but are part of the answer; this introduces a number of parameters which must be determined as part of the overall solution. The most important of these are T^*, the air–water interface temperature, z^*, the water layer thickness, and w_m, the melting rate (or the non-dimensional counterparts of these quantities).

In the actual computation, families of solutions were found corresponding to a range of these parameters and the appropriate member of the family determined by matching the solutions at the interfaces.

The only truly arbitrary parameters in the problem are T_∞, $T_{-\infty}$ and β_1, which characterize the outside potential flow of the air ($u \sim \beta_1 z$ as $z \to \infty$), and the solution must be expressed in terms of these three quantities. In some sections of the paper it is easier and more convenient to express the solution in terms of T^* rather than T_∞ but all results are given eventually in terms of T_∞, $T_{-\infty}$ and β_1.

The solution of the equation for heat transfer in the melting ice is first obtained in simple exact form, in terms of the unknown melting rate w_m. Attention is then turned to the equations which govern the motion of the air. These are reduced to a system of two ordinary differential equations of which an approximate solution is obtained involving two unknown quantities, T^* and ϵ (a small parameter occurring in the velocity at the air–water interface).

The equations for the water layer are also reduced to ordinary differential equations and it is shown that the number of boundary conditions to be satisfied at $z = 0$ and $z = z^*$ are just sufficient to determine the solution in the water layer and all the unknown parameters that are introduced.

Simple estimates are then deduced, without solving the equations for motion in the water layer, which give useful approximate results concerning the rate of melting and the shielding effect of the water. Finally, a better approximation which uses the Pohlhausen method to determine the water layer thickness and the approximate velocity and temperature profiles in the water is made; this also gives a check on the first simple estimates.

3. Heat transfer in the ice

The transfer of heat in a solid moving with constant velocity w_m (unknown) in the z-direction is governed by

$$\rho_3 c_3 w_m \frac{\partial T}{\partial z} = \frac{\partial}{\partial x}\left(k_3 \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z}\left(k_3 \frac{\partial T}{\partial z}\right); \tag{3.1}$$

the boundary conditions for the present problem are

$$\left.\begin{aligned} T &= T_m \qquad \text{at } z = 0, \\ T &= T_{-\infty} \qquad \text{as } z \to -\infty, \end{aligned}\right\} \tag{3.2}$$

where the quantities ρ_3, c_3, k_3 and T_m are those defined in § 2.2.

The solution of (3.1) which is a function of z only and satisfies (3.2) is obtained by elementary methods and is written as

$$T = T_\infty + (T_m - T_{-\infty})\exp\left(\frac{w_m \rho_3 c_3}{k_3}\right). \tag{3.3}$$

This gives the rate of heat transfer at $z = 0$ to the interior of the ice,

$$\left(k_3 \frac{\partial T}{\partial z}\right)_{z=0} = w_m \rho_3 c_3 (T_m - T_{-\infty}); \tag{3.4}$$

equation (3.4) will be required later in the formulation of the boundary conditions, at $z = 0$, for the equations which describe the motion of the water.

The form of the solution (3.3) shows that there is a thermal boundary layer near the surface $z = 0$ in which the temperature changes from T_0 to $T_{-\infty}$; the thickness of this thermal layer is $O(k_3/w_m \rho_3 c_3)$ and thus varies inversely with the rate of melting and directly with the thermal diffusivity, $k_3/\rho_3 c_3$.

4. The motion of the air

4.1. *The equations and the boundary conditions*

Consider the steady flow of air of variable density ρ_1, viscosity μ_1, and thermal conductivity k_1, at zero Mach number in two dimensions. If the components of velocity in the x- and z-directions are u and w respectively, and the pressure is p, then the equations of continuity, momentum, and energy are

$$\frac{\partial(\rho_1 u)}{\partial x}+\frac{\partial(\rho_1 w)}{\partial z}=0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_1}\frac{\partial p}{\partial x}+\frac{1}{\rho_1}\frac{\partial}{\partial x}\left\{2\mu_1\frac{\partial u}{\partial x}-\frac{2}{3}\mu_1\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right\}+ \\ +\frac{1}{\rho_1}\frac{\partial}{\partial z}\left\{\mu_1\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right\}, \tag{4.2}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_1}\frac{\partial p}{\partial z}+\frac{1}{\rho_1}\frac{\partial}{\partial z}\left\{2\mu_1\frac{\partial w}{\partial z}-\frac{2}{3}\mu_1\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right\}+ \\ +\frac{1}{\rho_1}\frac{\partial}{\partial x}\left\{\mu_1\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right\}, \tag{4.3}$$

$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\frac{1}{\rho_1\sigma_1}\left\{\frac{\partial}{\partial x}\left(\mu_1\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu_1\frac{\partial T}{\partial z}\right)\right\}, \tag{4.4}$$

where $c_{p1}=$ constant and $\sigma_1=\mu_1 c_{p1}/k_1=$ constant are respectively the specific heat at constant pressure and the Prandtl number, and dissipation has been neglected*.

At large distances from the body the flow has the potential form

$$u=\beta_1 x, \qquad w=-\beta_1 z, \qquad T=T_\infty,$$

so that the boundary conditions for the problem under consideration are:

$$\left.\begin{array}{lll} u=\beta_1\epsilon x, \quad w=0, & T=T^*, & \text{at } z=z^*; \\ u\sim\beta_1 x, & T\to T_\infty, & \text{as } z\to\infty. \end{array}\right\} \tag{4.5}$$

Here β_1 and T_∞ are given by the potential flow; T^* and ϵ (a small parameter which represents the effect of the water layer on the motion of the air) are unknown and will be determined by the rate at which the ice melts.

It is well-known that the Navier–Stokes equations for incompressible flow have an exact solution which represents flow near a stagnation point. This solution, the Hiemenz solution (Goldstein 1938, vol. 1, pp. 139–140), has the form

$$u=\beta_1 x f'(\eta), \qquad w=-(\beta_1 u_1/\rho_1)^{1/2}f(\eta),$$

where

$$\eta=(\beta_1\rho_1/\mu_1)^{1/2}z,$$

* The terms $\mu/\rho c_p(\partial u/\partial z)^2$ and $u/\rho c_p\,\partial p/\partial x$ which represent the change in temperature due respectively to the heat generated by skin friction and the adiabatic compression are $O((\gamma-1)M^2)$ compared with other terms in equation (4.4), and have been neglected. A comparison of the results of Levy & Seban (1953) and Brown & Donoghe (1951) shows that this neglect is justified.

and f satisfies the differential equation

$$f'^2 - ff'' = 1 + f''' \tag{4.6}$$

and the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1 \tag{4.7}$$

(a dash denotes differentiation with respect to η).

In this solution the fluid is assumed to have constant properties so that the temperature, $T = T_0 + (T_\infty - T_0)g$ (T_0 = wall temperature) may be calculated separately from the reduced energy equation

$$\left.\begin{aligned} &g'' + \sigma_1 f g' = 0, \\ \text{with} \qquad &g(0) = 0, \quad g(\infty) = 1. \end{aligned}\right\} \tag{4.8}$$

For such a flow the boundary layer has constant thickness and the vertical component of velocity and the temperature are constant in layers at a uniform distance from the wall.

In an analogous way we seek a solution which has similar properties but which takes account of the effect of the temperature field on the density and viscosity (through the assumptions of § 2.2) and consequently on the velocity field.

We consider a solution of the form

$$\left.\begin{aligned} &u = \beta_1 x f_1'(\eta_1), \qquad \frac{\rho}{\rho_\infty} w = -(\beta_1 \mu_\infty/\rho_\infty)^{1/2} f_1(\eta_1), \\ &\frac{T}{T_\infty} = g_1(\eta_1), \qquad \int_0^{\eta_1} \frac{\rho_\infty}{\rho} d\eta_1 = (\beta_1 \rho_\infty/\mu_\infty)^{1/2} (z - z^*), \end{aligned}\right\} \tag{4.9}$$

where $\rho_1 T = \rho_\infty T_\infty$, and $\mu_1/T = \mu_\infty/T_\infty$. With new dependent variables $f_1(\eta_1)$, $g_1(\eta_1)$ and the new independent variable η_1, (4.1) is satisfied, (4.2) becomes

$$f_1'^2 - f_1 f_1'' = g_1 + f_1''', \tag{4.10}$$

and (4.4) becomes

$$g_1'' + \sigma_1 f_1 g_1' = 0, \tag{4.11}$$

where a dash now denotes differentiation with respect to η_1.

The pressure p, is given by

$$p = -\tfrac{1}{2}\rho_\infty \beta_1^2 x^2 - \pi(\eta_1)\beta_1 \mu_\infty,$$

where, from (4.3), $\pi(\eta_1)$ satisfies the reduced momentum equation

$$\pi' = \{f_1(f_1 g_1)' + [\tfrac{4}{3}(f_1 g_1)' + \tfrac{2}{3} g_1 f_1']' - g_1 f_1''\}. \tag{4.12}$$

The boundary conditions (4.5) reduce to:

$$f_1(0) = 0, \quad f_1'(0) = \epsilon, \quad g_1(0) = \frac{T^*}{T_\infty}, \quad f_1'(\infty) = 1, \quad g_1(\infty) = 1. \tag{4.13}$$

An approximate solution of (4.10) and (4.11) with the boundary conditions (4.13) is obtained by considering a perturbation, in ϵ, of the particular solution for which $\epsilon = 0$, i.e.

$$f_1 = \phi_0 + \epsilon\phi_1 + O(\epsilon^2), \quad \text{and } g_1 = \psi_0 + \epsilon\psi_1 + O(\epsilon^2). \tag{4.14}$$

The solution (ϕ_0, ψ_0) is the first approximation and takes no account of the effect of the motion of the water.

4.2. *The initial approximation*, $\epsilon = 0$

When $\epsilon = 0$, the boundary conditions (4.13) are those for flow past a solid boundary; the approximate solution of the system (4.10), (4.11) with (4.13) has been given by Levy & Seban (1953); it was assumed that the solution ϕ_0, ψ_0 has the approximate form

$$\left.\begin{aligned} \phi_0' &= 1-\exp\left(a\eta_1+\frac{b}{2\,!}\eta_1^2+\frac{c}{3\,!}\eta_1^3+\frac{d}{4\,!}\eta_1^4\right), \\ \psi_0 &= 1-K\exp\left(A\eta_1+\frac{B}{2\,!}\eta_1^2+\frac{C}{3\,!}\eta_1^3+\frac{D}{4\,!}\eta_1^4\right), \end{aligned}\right\} \tag{4.15}$$

so that the boundary conditions as $\eta_1 \to \infty$ are satisfied if d and D are negative; the non-dimensional temperature at $\eta_1 = 0$ is $\psi_0(0) = (1-K)$. The constants a, b, c, d, and A, B, C, D, are determined by satisfying the boundary conditions and the differential equations as far as possible at $\eta_1 = 0$. The system of algebraic equations which results reduces to the following pair which give a and A in terms of K

$$\begin{aligned} 10KA[2a^2-(1-K)] &= 24a^5-5a^3[12(1-K)-1]+ \\ &\qquad +2a[15(1-K)^2-(1-K)], \\ 24A^5-5\sigma_1 aA^2 &= -\sigma_1(1-K)A; \end{aligned} \tag{4.16}$$

the remaining unknown coefficients are then easily expressed in terms of K.

The solution $a(K)$, $A(K)$, of (4.16) has been tabulated for some values of K; further work was required in an extended range for K, for the needs of the present paper. (The quantities $-a = \phi_0''(0)$ and $-A = \psi_0'(0)$ are important since they appear in the expressions for the skin friction and the rate of heat transfer respectively at $\eta_1 = 0$.)

4.3. *Flow past the slowly-moving layer of water*

The functions ϕ_1, ψ_1 which are required for the small perturbation from ϕ_0, ψ_0 satisfy the linear equations

$$\left.\begin{aligned} \phi_1'''+\phi_0\phi_1''-2\phi_0'\phi_1'+\phi_0''\phi_1+\psi_1 &= 0, \\ \psi_1''+\sigma_1\phi_0\psi_1'+\sigma_1\psi_0'\phi_1 &= 0, \end{aligned}\right\} \tag{4.17}$$

where terms of $O(\epsilon^2)$ and higher have been neglected. The solution of (4.17), chosen so that the total solution satisfies the boundary conditions (4.13), is

$$\phi_1 = -a^{-1}\phi_0', \qquad \psi_1 = -a^{-1}\psi_0',$$

and the total solution is

$$f_1 = \phi_0-\epsilon a^{-1}\phi_0', \qquad g_1 = \psi_0-\epsilon a^{-1}\psi_0', \tag{4.18}$$

from which the relation between T^* and K is obtained,

$$\frac{T^*}{T_\infty} = 1-K\left(1-\frac{\epsilon A}{a}\right). \tag{4.19}$$

4.4. *Important physical quantities evaluated at the air–water interface*

Before the motion of the water layer is considered it is necessary to evaluate the components of velocity, the components of stress, the rate of heat transfer, and the temperature, at the air–water interface $z = z^*$ (or $\eta_1 = 0$), since these quantities determine the amount of water produced and the manner in which it flows.

At $z = z^*$, the horizontal and vertical components of velocity are respectively

$$u^* = \beta_1 \epsilon x, \qquad w^* = 0; \tag{4.20}$$

the horizontal and vertical components of stress are respectively

$$\begin{aligned} \tau^* &= \mu_1\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ &= x\beta_1^{3/2}(-a)(\rho_1\mu_1)^{1/2}\left\{1 - \frac{\epsilon}{a^2}(1-K)\right\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \nu^* &= -p + \mu_1\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right\} \\ &= \tfrac{1}{2}\rho_\infty \beta_1^2 x^2 + \beta_1 \mu_\infty\{\pi + (2g_1 f_1' + \tfrac{2}{3} f_1 g_1')\}_{\eta_1 = 0}. \end{aligned} \tag{4.22}$$

(It is not necessary to evaluate the expression in curly brackets explicitly unless the distribution of stress along the air–water interface is required.) The rate of transfer of heat is

$$q^* = -k\frac{\partial T}{\partial z} = KA\beta_1^{1/2}(\rho_1\mu_1)^{1/2}\frac{c_{p1}}{\sigma_1}T_\infty, \tag{4.23}$$

and the temperature is

$$T^* = T_\infty\left[1 - K\left(1 - \epsilon\frac{A}{a}\right)\right]. \tag{4.24}$$

In the above expressions, β_1 and T_∞ are parameters prescribed by the potential flow, whereas ϵ and T^* (and therefore, K, a, and A) are quantities which must be determined as part of the solution of the whole problem.

The first shielding effect due to the raising of the air–water interface temperature to T^* may be written as the ratio,

$$\begin{aligned} &\frac{\text{Rate of heat transfer from the air with water layer}}{\text{Rate of heat transfer from the air without water layer}} \\ &= K\left(\frac{T^*}{T_\infty}\right)A\left(\frac{T^*}{T_\infty}\right)\Big/K\left(\frac{T_m}{T_\infty}\right)A\left(\frac{T_m}{T_\infty}\right) \\ &= \frac{T_\infty - T^*}{T_\infty - T_m}A\left(\frac{T^*}{T_\infty}\right)\Big/A\left(\frac{T_m}{T_\infty}\right)\left(1 - \epsilon\frac{A}{a}\right). \end{aligned} \tag{4.25}$$

When T_∞ is large compared with T^* and T_m, (4.25) may be approximated by $(T_\infty - T^*)/(T_\infty - T_m)$ since $A(T/T_\infty)$ given by (4.16) is very nearly constant and $\epsilon A/a$ is very small.

The ratio $(T_\infty - T^*)/(T_\infty - T_m)$ is determined from the final solution to the whole problem in a later section.

5. THE WATER LAYER

With the expressions (3.4) and (4.20) to (4.24), which determine the rate of production and the subsequent motion of the water, we are now in a position to consider the equations of motion and the boundary conditions for the water layer.

5.1. *The equations of motion*

The equations which govern the motion of, and the transfer of heat in, the layer of water of constant density ρ_2, thermal conductivity k_2, and specific heat c_2, but variable viscosity μ_2, are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{5.1}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_2}\frac{\partial p}{\partial x}+\frac{1}{\rho_2}\frac{\partial}{\partial x}\left(\mu_2\frac{\partial u}{\partial x}\right)+\frac{1}{\rho_2}\frac{\partial}{\partial z}\left(\mu_2\frac{\partial u}{\partial z}\right), \tag{5.2}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_2}\frac{\partial p}{\partial z}+\frac{1}{\rho_2}\frac{\partial}{\partial x}\left(\mu_2\frac{\partial w}{\partial x}\right)+\frac{1}{\rho_2}\frac{\partial}{\partial z}\left(\mu_2\frac{\partial w}{\partial z}\right), \tag{5.3}$$

$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\frac{k_2}{\rho_2 c_2}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right), \tag{5.4}$$

where dissipation has been neglected.

In order to satisfy the continuity of the expressions (4.20) to (4.24), at $z=z^*$ we consider a solution of the following form:

$$\left.\begin{aligned} u &= \beta_2 x f_2'(\eta_2), & w &= -(\mu_{2m}\beta_2/\rho_2)^{1/2} f_2(\eta_2),\\ T &= T_m+(T^*-T_m)g_2(\eta_2), & \eta_2 &= (\beta_2\rho_2/\mu_{2m})^{1/2} z, \end{aligned}\right\} \tag{5.5}$$

where $\mu_{2m}=\mu_2$ at $T=T_m$, and β_2 is a parameter which must be determined from the boundary conditions at the air–water interface $z=z^*$. A dash denotes differentiation with respect to η_2.

With dependent variables $f_2(\eta_2)$ and $g_2(\eta_2)$, and the independent variable η_2, equation (5.1) is satisfied, (5.2) becomes

$$\left(\frac{\mu_2}{\mu_{2m}}f_2''\right)'+1=f_2'^2-f_2 f_2'', \tag{5.6}$$

and (5.4) becomes

$$g_2''+\sigma_{2m}f_2 g_2'=0. \tag{5.7}$$

The second momentum equation (5.3) may be integrated to give

$$p+\tfrac{1}{2}(\rho_2\beta_2 x^2+\rho_2 w^2)-\mu_2\frac{\partial w}{\partial z}=\text{constant}. \tag{5.8}$$

5.2. *The boundary conditions*

(i) At the air–water interface

The quantities, u^*, w^*, τ^*, ν^*, g^* and T^* given by the solution of (5.6) and (5.7) must be equal to the expressions for these quantities given by (4.20) to (4.24); i.e. if the air–water interface is given by

$$\eta_2=\eta_2^*=(\beta_2\rho_2/\mu_{2m})^{1/2}z^*,$$

then at $\eta_2=\eta_2^*$,

$$\beta_2 f_2'(\eta_2^*)=\beta_1\epsilon, \qquad f_2(\eta^*)=0, \tag{5.9}$$

$$\beta_2^{3/2}(\rho_2\mu_{2m})^{1/2}\frac{\mu_2^*}{\mu_{2m}}f_2''(\eta_2^*) = (-a)\beta_1^{3/2}(\rho_1\mu_1)^{1/2}\left[1-\frac{\epsilon}{a^2}(1-K)\right]. \quad (5.10)$$

A comparison of terms involving x^2 in (4.2) and (5.8) gives

$$\tfrac{1}{2}\rho_2\beta_2^2 = \tfrac{1}{2}\rho_\infty\beta_1^2. \quad (5.11$$

The heat transfer condition is

$$\beta_2^{1/2}(\rho_2\mu_{2m})^{1/2}\frac{c_2}{\sigma_{2m}}\frac{T^*-T_m}{T_\infty}g_2'(\eta_2^*) = (-A)\beta_1^{1/2}(\rho_1\mu_1)^{1/2}\frac{c_{p1}}{\sigma_1}K, \quad (5.12)$$

and the temperature $T = T^*$ gives

$$g_2(\eta^*) = 1. \quad (5.13)$$

The small parameter ϵ is found by elimination of β_2/β_1 from (5.9) and (5.11), i.e.

$$\epsilon = f_2'(\eta_2^*)(\rho_\infty/\rho_2)^{1/2}. \quad (5.14)$$

Numerical values of ϵ show that it is $O(10^{-2})$ so that the neglect of quantities of $O(\epsilon^2)$ in § 4 is justified.

(ii) At the melting surface

There are three boundary conditions to be satisfied at the melting surface $z = 0$; these are:

(*a*) the mass of water introduced is equal to the amount of the ice lost by the body,

(*b*) the heat transferred from the water is equal to that transferred to the interior of the ice plus the latent heat of melting,

(*c*) water is produced at the melting temperature T_m.

The boundary conditions (*a*) and (*b*) are most easily formulated in the following way. The ice melts at a constant rate w_m so that, for unit surface area, in unit time,

$$\text{mass of water produced} = w_m\rho_3$$

and,
$$\text{volume of water produced} = w_m\rho_3/\rho_2;$$

thus the equivalent velocity of the water at the melting surface is

$$w_2(0) = w_m\rho_3/\rho_2, \quad (5.15)$$

and since the heat transfer and thus the melting take place in the z-direction and are independent of x, we also have

$$u_2(0) = 0. \quad (5.16)$$

The heat transfer condition is written

$$k_2\left(\frac{\partial T}{\partial z}\right)_{z=0+} - k_3\left(\frac{\partial T}{\partial z}\right)_{z=0-} = Lw_m\rho_3 = Lw_2(0)\rho_2, \quad (5.17)$$

and since
$$k_3\left(\frac{\partial T}{\partial z}\right)_{z=0-} = c_3(T_m - T_{-\infty})w_m\rho_3 \quad \text{(from (3.4))},$$

we have
$$k_2\left(\frac{\partial T}{\partial z}\right)_{y=0+} = [L + c_3(T_m - T_{-\infty})]w_2(0)\rho_2. \quad (5.18)$$

Equation (5.18) is a statement that in unit time the heat transfer $k_2(\partial T/\partial z)_{z=0+}$ causes the mass $w_2(0)\rho_2$ ($= w_m \rho_3$) of ice to be raised through a temperature difference $T_m - T_{-\infty}$ and then supplies the heat to melt it.

In terms of f_2, g_2 and η_2, (5.16) and (5.18) are

$$f_2'(0) = 0, \tag{5.19}$$

$$g_2'(0) = -\frac{L + c_3(T_m - T_{-\infty})}{c_2(T^* - T_m)}\sigma_{2m} f_2(0), \tag{5.20}$$

and the condition (c) gives

$$g_2(0) = 0. \tag{5.21}$$

Equation (5.15) is used to calculate w_m when $w_2(0)$ has been determined as part of the solution.

A solution of the equations (5.6) and (5.7) is required subject to the nine boundary conditions (5.9) to (5.13), (5.19) to (5.21); these conditions are just sufficient to obtain a unique solution and determine the parameters T^*, ϵ, η_2^* and β_2 (for given quantities T_∞, $T_{-\infty}$, and β_1) when K, A, and a are expressed in terms of T^* and ϵ through (4.24) and (4.16).

Clearly, the exact calculations of this system of differential equations and boundary conditions would be complicated. In the following section, simple approximate relations which give a good overall picture of the inter-relation among the various parameters are derived and the results are checked by applying a second, more accurate, approximate method.

6. The shielding effect and the approximate methods of solution

The important features of the flow in the water layer are the rate of convection of mass, which is related to the rate of melting of the ice, and the rate of convection of heat which causes a reduction in the rate of heat transfer to the ice; this, in turn, controls the rate of melting. In the steady state the rate of melting is such that these effects balance.

The important non-dimensional quantity that determines the rate of melting presents itself when (5.20) is written in the form

$$-f_2(0) = \frac{c_2(T^* - T_m)}{L + c_3(T_m - T_{-\infty})}\frac{1}{\sigma_{2m}} g_2'(0). \tag{6.1}$$

It is seen that the parameter

$$\frac{c_2(T^* - T_m)}{L + c_3(T_m - T_{-\infty})} = P$$

is significant in the determination of $f_2(0)$ (which is related to the rate of melting) in terms of $g_2'(0)$ (essentially the rate of heat transfer to the ice). Although P contains the unknown temperature T^* it is convenient to express all quantities in terms of P wherever possible and thus obtain implicit relations from which T^* can be eliminated finally.

A simple expression which describes the shielding effect of the water layer is found in the following way. Equation (5.7) is integrated between the limits 0 and η_2^* to give a heat balance equation

$$g_2'(\eta^*) - g_2'(0) = -\sigma_{2m} \int_0^{\eta_2^*} f_2\, g_2'\, d\eta_2$$

$$= \sigma_{2m} \int_0^{\eta_2^*} f_2'\, g_2\, d\eta_2, \tag{6.2}$$

since $g_2(0) = 0$ and $f_2(\eta^*) = 0$. Equation (6.1) is written

$$g_2'(0) = \frac{\sigma_{2m}}{P}(f_2(\eta_2^*) - f_2(0)). \tag{6.3}$$

Thus

$$\frac{g_2'(\eta_2^*) - g_2'(0)}{g_2'(0)} = \bar{g}_2 P \tag{6.4}$$

where

$$\bar{g}_2 = \int_0^{\eta_2^*} f_2'\, g_2\, d\eta_2 \bigg/ \int_0^{\eta_2^*} f_2'\, d\eta_2. \tag{6.5}$$

Since $x \int_0^{\eta_2^*} f_2' g_2\, d\eta_2$ and $x \int_0^{\eta_2^*} f_2'\, d\eta_2$ are respectively the rates of transfer of heat and mass in the x-direction by convection, $\bar{g}_2$ is seen to be the mean non-dimensional temperature to which the water is raised during convection. A measure of the shielding effect is given by the ratio

$$\frac{\text{Rate of transfer of heat from water to ice}}{\text{Rate of transfer of heat from air to water}}$$

$$= \frac{g_2'(0)}{g_2'(\eta_2^*)} = \frac{1}{1 + \bar{g}_2 P} = R_2. \tag{6.6}$$

The rate of melting may be expressed in terms of q^* and T^* by elimination of $g_2'(0)$ from (6.3) and (6.6):

$$g_2'(\eta^*) = \frac{1 + \bar{g}_2 P}{P} \sigma_{2m}(-f_2(0)),$$

which gives

$$q^* = c_3(T_m - T_{-\infty}) + \{L + \bar{g}_2 c_2(T^* - T_m)\} w_m \rho_3. \tag{6.7}$$

Equation (6.7) states that in unit time, for unit area of ice surface, the heat q^* transferred across the air–water interface first raises the temperature of the mass $w_m \rho_3$ of ice from $T_{-\infty}$ to T_m, melts it and raises the temperature of the water produced from T_m to the average temperature $T_m + \bar{g}_2(T^* - T_m)$ during convection.

6.1. *Simple estimates*

Simple expressions may be obtained for the shielding ratio R_2, the rate of melting, and the thickness of the water layer and the thermal boundary layer in the ice.

We assume that the effect of the motion of the water on the heat transfer and stress at the interface are negligible so that quantities of order ϵ are ignored when terms of order unity are present. It is also assumed that the main features of the flow do not depend critically on the local velocity and temperature profiles which are given the forms

$$\left.\begin{aligned} f_2 &= f_2(0)\left(1-\frac{\eta_2^2}{\eta_2^{*2}}\right), \\ g_2 &= \alpha\,\frac{\eta_2}{\eta_2^*}+(1-\alpha)\,\frac{\eta^2}{\eta_2^{*2}}; \end{aligned}\right\} \tag{6.8}$$

these expressions satisfy the boundary conditions

$$f_2'(0) = 0, \quad f_2'(\eta^*) = 0; \qquad g_2(0) = 0, \quad g_2(\eta^*) = 1.$$

The average non-dimensional temperature $\bar{g}_2$ is expressed in terms of α, by

$$\bar{g}_2 = \tfrac{1}{2}+\tfrac{1}{6}\alpha,$$

and the shielding ratio

$$\frac{g_2'(0)}{g_2'(\eta_2^*)} = \frac{1}{1+\bar{g}_2 P} = \frac{\alpha}{2-\alpha} = R_2,$$

so that α is given by the positive root of the equation

$$P\alpha^2+(12+3P)\alpha-12 = 0. \tag{6.9}$$

The melting condition (6.1) and the air–water interface stress condition, simplified by (6.8), are

$$-f_2(0) = \frac{P}{\sigma_{2m}}\frac{\alpha}{\eta_2^*} \tag{6.10}$$

and

$$-f_2(0)\frac{2}{\eta_2^{*2}}\frac{\mu_2^*}{\mu_{2m}} = (-a)\left(\frac{\beta_1}{\beta_2}\right)^{3/2}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{1/2}, \tag{6.11}$$

which give

$$-\left(\frac{\beta_2}{\beta_1}\right)^{1/2} f_2(0) = \left(\frac{P\alpha}{\sigma_{2m}}\right)^{2/3}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{1/6}\left(-\tfrac{1}{2}a\frac{\mu_{2m}}{\mu_2^*}\right)^{1/3} \tag{6.12}$$

and

$$\left(\frac{\beta_1}{\beta_2}\right)^{1/2}\eta_2^* = \left(\frac{P\alpha}{\sigma_{2m}}\right)^{1/3}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{-1/6}\left(-\tfrac{1}{2}a\frac{\mu_{2m}}{\mu_2^*}\right)^{-1/3}. \tag{6.13}$$

Equations (6.12) and (6.13) express $f_2(0)$ and η_2^* in terms of an unknown parameter T^* (through P and α). However, the heat transfer condition (5.12) is used to express $P(T^*)$ in terms of a basic parameter

$$\frac{c_{p1}(T_\infty - T_m)}{L+c_3(T_m-T_{-\infty})} = S. \tag{6.14}$$

We have, from (5.12), (4.24), (6.8) and (6.12)

$$\frac{T_\infty - T^*}{T^*-T_m} = \left(\frac{a}{2A^3}\right)^{1/3}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{-1/3}\frac{c_2}{c_{p1}}\left(\frac{\mu_{2m}}{\mu_2^*}\right)^{1/3}\sigma_1\left(\frac{P\alpha}{\sigma_{2m}}\right)^{2/3}\frac{1+\bar{g}_2 P}{P},$$

and since

$$\frac{T_\infty - T^*}{T^*-T_m} = \frac{T_\infty - T_m}{T^*-T_m}-1 = \frac{c_{p1}(T_\infty - T_m)}{L+c_3(T_m-T_{-\infty})}\frac{c_2}{Pc_{p1}}-1,$$

then S is related to P by

$$S = \frac{c_{p1}}{c_2}P+\left(\frac{a}{2A^3}\right)^{1/3}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{-1/3}\left(\frac{\mu_{2m}}{\mu_2^*}\right)^{1/3}\left(\frac{P\alpha}{\sigma_{2m}}\right)^{2/3}\sigma_1(1+\bar{g}_2 P). \tag{6.15}$$

The quantity A varies only very slowly over a large range of the temperature ratio T^*/T and an average value is taken. Average values of μ_2^*/μ_{2m} and a are also used. The factor $(\rho_1\mu_1/\rho_2\mu_{2m})^{-1/3}$ depends on the stagnation pressure but has been assumed independent of temperature throughout this paper.

Equation (6.15) is used in order to plot quantities of interest as functions of S rather than of P. The rate of melting $w_m = w(0)\rho_2/\rho_3$ is given in non-dimensional form by

$$w_m\left(\frac{\rho_1}{\mu_1\beta_1}\right)^{1/2} = \left(\frac{P\alpha}{\sigma_{2m}}\right)^{2/3}\frac{\rho_1}{\rho_3}\left(\frac{\rho_2\mu_{2m}}{\rho_1\mu_1}\right)^{1/3}\left(-\tfrac{1}{2}a\frac{\mu_{2m}}{\mu_2^*}\right)^{1/3}, \tag{6.16}$$

and the thickness of the water layer by

$$z^*\left(\rho_1\frac{\beta_1}{\mu_1}\right)^{1/2} = \left(\frac{P\alpha}{\sigma_{2m}}\right)^{1/3}\left(-\tfrac{1}{2}a\frac{\mu_{2m}}{\mu_2^*}\right)^{-1/3}\frac{\rho_1}{\rho_2}\left(\frac{\rho_2\mu_{2m}}{\rho_1\mu_1}\right)^{2/3}. \tag{6.17}$$

When the average values of μ_2^*/μ_{2m}, A, a, are inserted together with appropriate values of ρ_1, ρ_2, ρ_3 etc. (for air–water–ice) we have,

$$w_m\left(\frac{\rho_1}{\mu_1\beta_1}\right)^{1/2} = 0\cdot96\times10^{-2}(P\alpha)^{2/3} \tag{6.18}$$

and

$$z^*\left(\frac{\rho_1\beta_1}{\mu_1}\right)^{1/2} = 1\cdot22(P\alpha)^{1/3}, \tag{6.19}$$

where P is related to S by

$$S = 0\cdot24P + 10\cdot91(P\alpha)^{2/3}(1+\bar{g}_2P). \tag{6.20}$$

It is seen from (6.18) that the rate of melting w_m is $O(10^{-2})$ times a typical vertical velocity in the air boundary layer, while (6.19) shows that the thickness of the water layer is of the same order as that of the air boundary layer.

The thickness θ of the thermal boundary layer in the ice may be defined by

$$\theta = \frac{1}{T_m - T_{-\infty}}\int_0^{-\infty}(T-T_{-\infty})\,dz = \frac{k_3}{w_m\rho_3c_3}. \tag{6.21}$$

The non-dimensional form is written

$$\theta\left(\frac{\beta_1\rho_1}{\mu_1}\right)^{1/2} = \frac{\rho_1k_3}{\rho_3\mu_1c_3}\left(\frac{\mu_1\beta_1}{\rho_1}\right)^{1/2}\frac{1}{w_m}. \tag{6.22}$$

When the numerical values are inserted into (6.22) and (6.15) is used, we have

$$\theta\left(\frac{\beta_1\rho_1}{\mu_1}\right)^{1/2} = 8\cdot69(P\alpha)^{-2/3}. \tag{6.23}$$

It is seen that, for ice, the thermal boundary layer is considerably larger than the air boundary layer.

The existence of a thermal boundary layer of this order of magnitude show that the present theory is valid for bodies of finite depth during that time when the depth is much greater than the boundary layer thickness.

It also explains, in part, why the larger meteors which enter the earth's atmosphere do not become uniformly heated to high temperatures but remain relatively cool except in a small region near the melting surface; the thermal boundary layer would also be present if vaporization or burning occurred since the character of the solution (3.1) for the temperature distribution in the body would be the same.

6.2. *The improved approximation*

As a check on the accuracy of the estimates made in § 6.1 an approximate solution of the system of equations (5.6) and (5.7) with the boundary conditions (5.9) to (5.13), (5.19) to (5.21) is found by the Pohlhausen method for particular values of T_∞ and $T_{-\infty}$ and a stagnation pressure of 1 atmosphere. In this method $\epsilon \neq 0$ so that account is taken of the effect of the motion of the water on that of the air.

Equations (5.6) and (5.7) are written in integrated form

$$\frac{\mu_2^*}{\mu_{2m}} f_2''(\eta_2^*) - f_2''(0) + \eta^* = 2\int_0^{\eta_2^*} f_2'^2\, d\eta_2, \tag{6.24}$$

$$g_2'(\eta_2^*) - g_2'(0) = \sigma_{2m}\int_0^{\eta_2^*} f_2'\, g_2\, d\eta_2. \tag{6.25}$$

The boundary conditions at $\eta_2 = \eta_2^*$ are

$$\left.\begin{aligned} f_2(\eta_2^*) &= 0, \qquad f_2'(\eta_2^*) = \frac{\beta_1}{\beta_2}\epsilon, \\ \frac{\mu_2^*}{\mu_{2m}} f_2''(\eta_2^*) &= -a\left(\frac{\beta_1}{\beta_2}\right)^{3/2}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{1/2}\left(1-\frac{\epsilon}{a^2}(1-K)\right), \\ g_2(\eta_2^*) &= 1, \\ g_2'(\eta_2^*) &= -A\left(\frac{\beta_1}{\beta_2}\right)^{1/2}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{1/2}\frac{c_{p1}(T_\infty - T^*)}{c_2(T^* - T_m)}\frac{\sigma_{2m}}{\sigma_1}\frac{1}{(1-\epsilon A/a)}, \end{aligned}\right\} \tag{6.26}$$

where

$$\frac{\beta_1}{\beta_2} = \left(\frac{\rho_2}{\rho_\infty}\right)^{1/2}, \qquad K = \left(1-\frac{T^*}{T_\infty}\right)\Big/\left(1-\frac{\epsilon A}{a}\right).$$

The boundary conditions at $\eta_2 = 0$ are

$$g_2(0) = 0, \qquad f_2'(0) = 0, \qquad g_2'(0) = -\frac{L + c_3(T_m - T_{-\infty})}{c_2(T^* - T_m)}\sigma_{2m} f_2(0). \tag{6.27}$$

This system of equations and boundary conditions was solved, with $T_{-\infty} = T_m$ (no conduction into the ice) for a range of values of T_∞ such that T^* took values from 273° K to 373° K, and for the single case $T_{-\infty} = 193$° K, $T_\infty = 4500$° K by assuming the following profiles

$$\left.\begin{aligned} f_2 &= f_{20} + f_{21}\frac{\eta_2}{\eta_2^*} + f_{22}\frac{\eta_2^2}{\eta_2^{*2}} + f_{23}\frac{\eta_2^3}{\eta_2^{*3}}, \\ g_2 &= g_{20} + g_{21}\frac{\eta_2}{\eta_2^*} + g_{22}\frac{\eta_2^2}{\eta_2^{*2}} + g_{23}\frac{\eta_2^3}{\eta_2^{*3}} \end{aligned}\right\} \tag{6.28}$$

(the number of parameters in (6.28) is one more than the number of relations available; the extra condition $g''(\eta_2^*) = 0$, from (5.7), was therefore used). The results obtained by this method are to be found in § 6.3.

6.3. *Discussion of the numerical results*

The first shielding effect due to displacement of the air by the water layer was found in § 4.4 as

$$\frac{\text{Rate of transfer from the air without the water layer}}{\text{Rate of transfer from the air with the water layer}} = \frac{T_\infty - T^*}{T_\infty - T_m}$$

approximately. This ratio expressed in terms of S is

$$\frac{T_\infty - T_m}{T_\infty - T^*} = 1 - \frac{P}{S}\frac{c_{p1}}{c_2} = R_1,$$

where P is given in terms of S through (6.20). The parameter P has the range $0 \leqslant P \leqslant 1{\cdot}25$ when the values $L = 80$ cal/gm, $c_2 = 1$ cal/gm °C, $T^*_{\max} = 100°$ C and $T_m = 0°$ C are used. The parameter S obtained from (6.20) has the range $0 \leqslant S \leqslant 18{\cdot}30$, the upper limit corresponding to $T^* = 100°$ C and no conduction of heat to the interior of the ice ($T_m - T_{-\infty}$). It is seen from the values of P and S that $R_1 = 1 - c_{p1}/c_2 P/S$ is always between 1·0 and 0·984 so that this shielding effect is a small one (less than 2%).

P	R_2		S	$\bar{g}_2$
	Simple estimate	Pohlhausen $T_{-\infty} = T_m$	Simple estimate	Simple estimate
0·125	0·923	0·923	2·904	0·660
0·25	0·859	0·857	4·841	0·654
0·50	0·756	0·748	8·345	0·644
0·75	0·675	0·663	11·710	0·635
1·00	0·613	0·594	15·041	0·627
1·25	0·568	0·538	18·300	0·620

Table 1.

The second shielding effect due to the convection of heat in the layer of water, given by

$$R_2 = \frac{\text{Rate of transfer of heat from water to ice}}{\text{Rate of transfer of heat from air to water}}$$

is by no means negligible as figure 2 shows.

The minimum value of R_2, corresponding to maximum convection of heat by the layer of water, is 0·538 so that approximately 46% of the heat transferred from the air is convected by the water. The good agreement between the results of the simple estimate with the values found by the Pohlhausen calculation (see table 1) suggests that R_2 is insensitive to the particular velocity and temperature profiles in the water layer; it was found that the mean temperature $\bar{g}_2$ varied little when the profiles were changed.

The other quantities of interest are the non-dimensional rate of melting $w_m(\rho_1/\mu_1\beta_1)^{1/2}$ shown in figure 3, the non-dimensional water layer thickness $z^*(\rho_1\beta_1/\mu_1)^{1/2}$ (figure 4) and the thermal boundary layer thickness $\theta(\rho_1\beta_1/\mu_1)^{1/2}$, (figure 5).

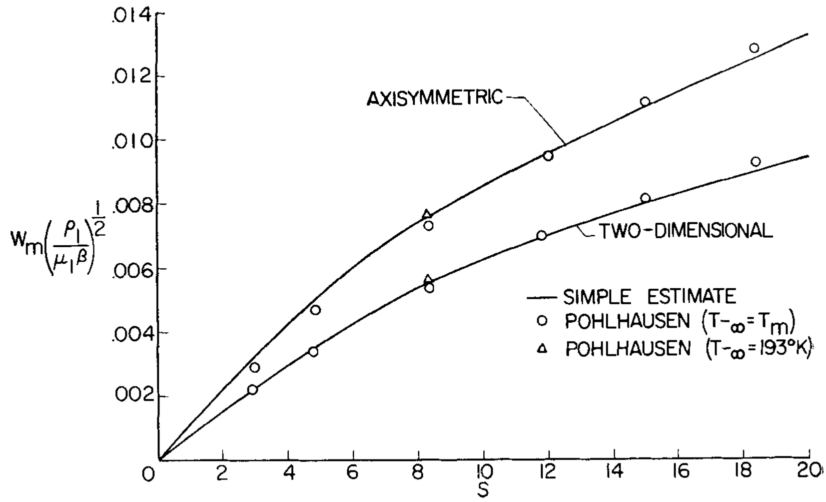

Figure 3. Graph of the melting rate.

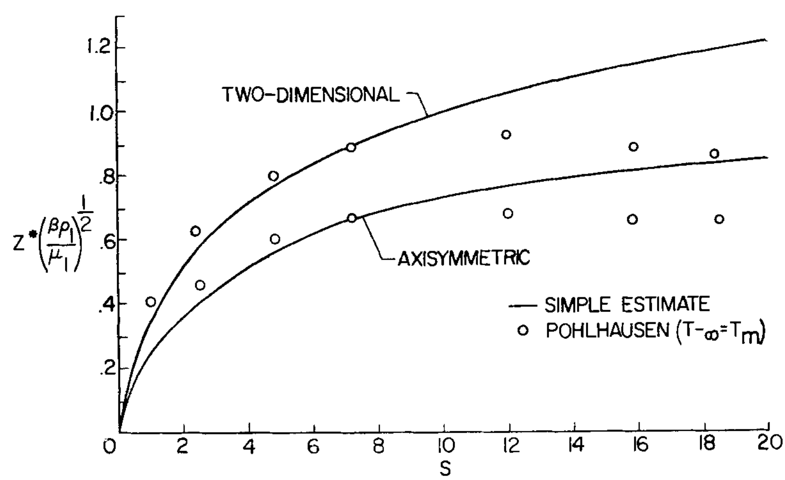

Figure 4. Graph of the water layer thickness.

The rate of melting w_m is $O(10^{-2})$ times a typical vertical velocity $(\mu_1\beta_1/\rho_1)^{1/2}$ in the air boundary layer, and the thickness of the water layer is of the same order as that of the air boundary layer. The good agreement between the simple estimates and the Pohlhausen calculation for the melting rate w_m is due in part to the fact that the quantity $(-\frac{1}{2}a\mu_{2m}/\mu_2^*)^{1/3}$, which involves average values, appears in the dominant term of the right-hand side of (6.15) and in (6.16) so that when $w_m(\rho_1/\mu_1\beta_1)^{1/2}$ is plotted against S the error caused by averaging is unimportant. The agreement between the simple estimate and the Pohlhausen calculation for the water layer thickness is not so good, however, since the averaged quantity appears as $(-\frac{1}{2}a\mu_{2m}/\mu_2^*)^{-1/3}$ in the expression (6.17) for $z^*(\rho_1\beta_1/\mu_1)^{1/2}$.

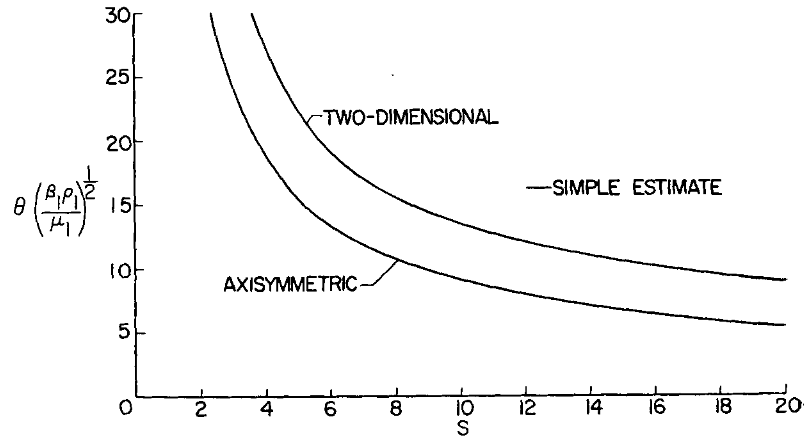

Figure 5. Graph of the thickness of the thermal layer in the ice.

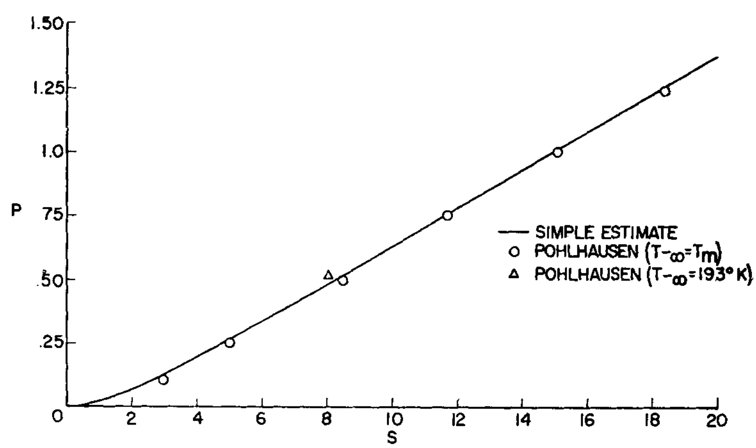

Figure 6. Graph of the relation between P and S.

The simple estimates indicate that the thickness of the water layer increases with S through the whole range. However, the Pohlhausen calculation, which takes some account of the variation of μ_2^*, shows that it is possible to have a maximum thickness. Due to the ability of the water to convect more easily at the higher temperatures (the viscosity of water decreases by a factor of 6·3 in the range 0° C to 100° C), the shielding effect due to convection increases even though the water thickness decreases slightly, as S increases. The temperature T^* is found from given values of T_∞ and $T_{-\infty}$ through

$$T^* = T_m + P[L + c_3(T_m - T_{-\infty})],$$

where P is found in terms of T_m and $T_{-\infty}$ through equations (6.20) or figure 6. (S is given in terms of T_∞ and $T_{-\infty}$.)

Figure 7 shows typical velocity and temperature profiles in the water layer as given by the Pohlhausen calculation.

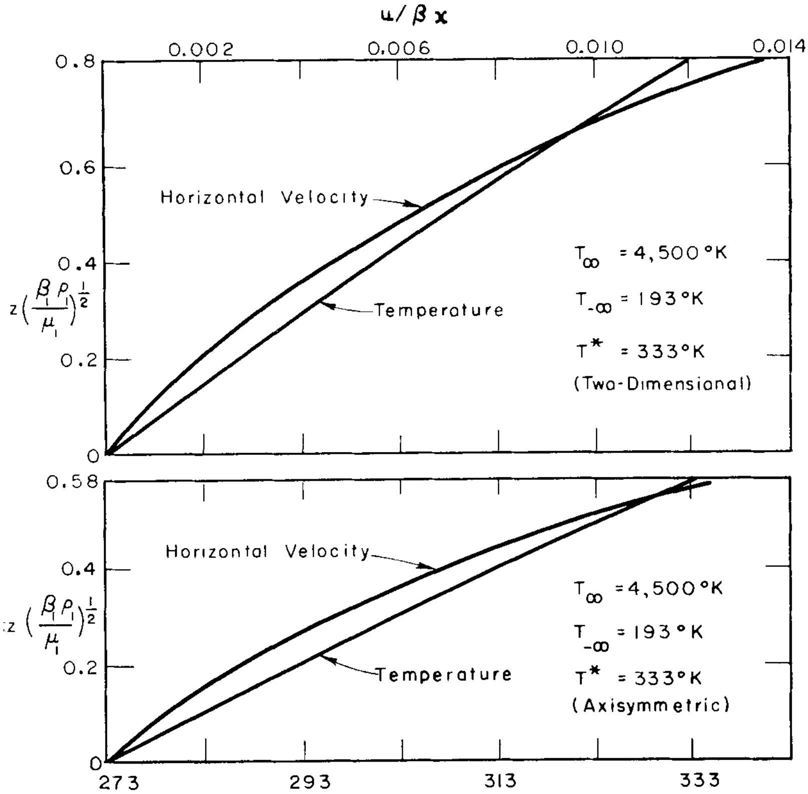

Figure 7. Temperature and velocity profiles.

7. The axisymmetric case

The methods described in the preceding sections may be applied, with only minor amendments, to the axisymmetric stagnation point flows. This is done briefly below.

7.1. *Heat transfer in the ice*

Equations (3.1) to (3.4) remain unaltered since they describe the heat transfer in the vertical direction. The solution is

$$T = T_{-\infty} + (T_m - T_{-\infty})\exp\left(\frac{w_m \rho_3 c_3}{k_3} z\right), \tag{7.1}$$

from which is obtained the rate of heat transfer from the melting surface to the interior.

$$\left(k_3 \frac{\partial T}{\partial z}\right)_{z=0-} = w_m \rho_3 c_3 (T_m - T_{-\infty}). \tag{7.2}$$

7.2. *The motion of the air*

At large distances from the body the flow has the potential form

$$u = \beta_1 x, \qquad w = -\beta_1 z, \qquad T = T_\infty, \tag{7.3}$$

where x is now the radial distance.

The non-dimensional variables f_1, g_1 and η_1 are now given by

$$\left.\begin{aligned} u = \beta_1 x f_1'(\eta_1), \qquad & \frac{\rho w}{\rho_\infty} = -\left(2\frac{\beta_1 \mu_\infty}{\rho_\infty}\right)^{1/2} f_1(\eta_1), \\ \frac{T}{T_\infty} = g_1(\eta_1) \quad \text{and} \quad & \int_0^{\eta_1} \frac{\rho_\infty}{\rho}\, d\eta_1 = \left(2\frac{\beta_1 \rho_\infty}{\mu_\infty}\right)^{1/2} (z - z^*). \end{aligned}\right\} \tag{7.4}$$

The equations of motion and energy for flows with axial symmetry become

$$\left.\begin{aligned} \tfrac{1}{2} f_1'^2 - f_1 f_1'' &= \tfrac{1}{2} g_1 + f_1''', \\ g_1'' + \sigma_1 f_1 g_1' &= 0, \end{aligned}\right\} \tag{7.5}$$

and the boundary conditions are

$$\left.\begin{aligned} f_1'(0) = 0, \qquad f_1'(0) = \epsilon, \qquad g_1(0) = T^*/T_\infty, \\ f_1'(\infty) = 1, \qquad g_1(\infty) = 1. \end{aligned}\right\} \tag{7.6}$$

The system (7.5) and (7.6) with $\epsilon = 0$ has been solved by the approximate method described in § 4.2. In the original solution (Levy & Seban 1953) (7.5) and (7.6) with $\epsilon = 0$ were shown to represent two-dimensional flow past a wedge; these equations together with (7.4) here represent axisymmetric stagnation point flow.

The solution $f_1 = \phi_0$, $g_1 = \psi_0$ has the same form (4.15) but now a and A are given by the simultaneous equations

$$\begin{aligned} 5KA[2a^2 - \tfrac{1}{2}(1-K)] &= 24a^5 - 5a^3[6(1-K)] + \\ &\qquad + a[\tfrac{15}{2}(1-K)^2 - \tfrac{1}{2}(1-K)], \end{aligned} \tag{7.7}$$

$$24A^5 - 5\sigma_1 a A^2 = -\tfrac{1}{2}\sigma_1(1-K)A.$$

The perturbed solution, for small ϵ, is again

$$f_1 = \phi_0 - \tfrac{1}{2}\epsilon\phi_0', \qquad g_1 = \psi_0 - \epsilon a^{-1}\psi_0', \tag{7.8}$$

and at the air–water interface we have the following expressions for the components of velocity and stress:

$$u^* = \beta_1 \epsilon x, \qquad w^* = 0, \tag{7.9}$$

$$\left.\begin{aligned} \tau^* &= x\beta_1^{3/2}(-a)(2\rho_1\mu_1)^{1/2}\left\{1-\frac{\epsilon}{2a^2}(1-K)\right\},\\ p^* &= \tfrac{1}{2}\rho_\infty\beta_1^2x^2+2\beta_1\mu_\infty\{\pi+(2g_1 f_1'+\tfrac{2}{3}f_1 g_1')\}_{\eta_1=0}, \end{aligned}\right\} \tag{7.10}$$

where π satisfies

$$\pi' = f_1(f_1 g_1)' + \{\tfrac{4}{3}(f_1 g_1)' + \tfrac{2}{3}g_1 f_1'\}' - g_1 f_1''. \tag{7.11}$$

The heat transfer and temperature are given by

$$\left.\begin{aligned} q^* &= KA(2\beta_1\rho_1\mu_1)^{1/2}\frac{c_{p1}}{\sigma_1}T_\infty,\\ T^* &= T_\infty\left[1-K\left(1-\frac{\epsilon A}{a}\right)\right]. \end{aligned}\right\} \tag{7.12}$$

7.3. *The motion of the water*

The appropriate non-dimensional variables are now given by

$$\left.\begin{aligned} u &= \beta_2 x f_2'(\eta_2), & w &= (2\mu_{2m}\beta_2/\rho_2)^{1/2}f_2(\eta_2),\\ T &= T_m + (T^*-T_m)g_2(\eta_2) \text{ and } & \eta_2 &= (2\beta_2\rho_2/\mu_{2m})^{1/2}z. \end{aligned}\right\} \tag{7.13}$$

The equations for axisymmetric incompressible flow with variable viscosity reduce to

$$\left.\begin{aligned} \left(\frac{\mu_2}{\mu_{2m}}f_2''\right)' + \tfrac{1}{2} &= \tfrac{1}{2}f_2'^2 - f_2 f_2'',\\ g_2'' + \sigma_{2m} f_2 g_2' &= 0. \end{aligned}\right\} \tag{7.14}$$

The boundary conditions at the air–water interface are

$$\left.\begin{aligned} &f_2(\eta_2^*) = 0, \qquad f_2'(\eta_2^*) = \frac{\beta_1}{\beta_2}\epsilon,\\ &\beta_2^{3/2}(\rho_2\mu_{2m})^{1/2}\frac{\mu_2^*}{\mu_{2m}}f_2''(\eta_2^*) = (-a)\beta_1^{3/2}(\rho_1\mu_1)^{1/2}\left\{1-\frac{\epsilon}{2a^2}(1-K)\right\},\\ &\beta_2^{1/2}(\rho_2\mu_{2m})^{1/2}\frac{c_2}{\sigma_{2m}}\frac{(T^*-T_m)}{T_\infty}g_2'(\eta_2^*) = (-A)\beta_1^{1/2}(\rho_1\mu_1)^{1/2}\frac{c_{p1}}{\sigma_1}K,\\ &\tfrac{1}{2}\rho_2\beta_2^2 = \tfrac{1}{2}\rho\beta_1^2, \qquad g_2(\eta_2^*) = 1, \end{aligned}\right\} \tag{7.15}$$

where

$$T^* = T_\infty\left[1-K\left(1-\frac{a}{\epsilon}A\right)\right]$$

and at the melting surface

$$\left.\begin{aligned} &f_2'(0) = 0, \qquad g_2(0) = 0,\\ \text{and}\qquad &g_2'(0) = -\frac{L+c_3(T_m-T_{-\infty})}{c_2(T^*-T_m)}\sigma_{2m}f_2(0). \end{aligned}\right\} \tag{7.16}$$

7.4. *The approximate solutions*

Simple estimates, similar to those obtained for the two-dimensional problem, are found by using the profiles

$$\left.\begin{aligned} f_2 &= f_2(0)\left(1-\frac{\eta_2^2}{\eta_2^{*2}}\right), \\ g_2 &= \alpha\,\frac{\eta_2}{\eta_2^*} - (1-\alpha)\frac{\eta_2^2}{\eta_2^{*2}}. \end{aligned}\right\} \tag{7.17}$$

The non-dimensional mean temperature $\bar{g}_2$ is unaltered, i.e.

$$\bar{g}_2 + \tfrac{1}{2} = \tfrac{1}{6}\alpha,$$

where

$$P\alpha^2 + (12+3P)\alpha - 12 = 0,$$

and P has the same meaning as before. The shielding ratio R_2 is the same as that in the two-dimensional problem (equation (6.6)). The rate of melting w_m, the water layer thickness, and the thermal boundary layer thickness in the ice, are given approximately by

$$w_m(\rho_1/\mu_1\,\beta_1)^{1/2} = 1{\cdot}33\times 10^{-2}(P\alpha)^{2/3},$$

$$z^*(\rho_1\,\beta_1/\mu_1)^{1/2} = 0{\cdot}88(P\alpha)^{1/3},$$

and

$$\theta(\rho_1\,\beta_1/\mu_1)^{1/2} = 6{\cdot}26(P\alpha)^{-2/3}$$

(average values of μ_2^*, a, and A have again been used).

The parameter P (which contains T^*) is again related to the basic parameter S by

$$S = \frac{c_{p1}}{c_2}P + \left(\frac{a}{2A^3}\right)^{1/3}\left(\frac{\rho_1\mu_1}{\rho_2\mu_{2m}}\right)^{-1/3}\left(\frac{\mu_{2m}}{\mu_2^*}\right)^{1/3}\left(\frac{P\alpha}{\sigma_{2m}}\right)^{2/3}(1+\bar{g}_2P).$$

A second calculation was carried out as a check on the accuracy of the results by assuming the velocity and temperature profiles (6.28) and applying the Pohlhausen method as described in § 6.2.

7.5. *Discussion of numerical results*

It is fortuitous that the quantity $(a/2A^3)^{1/3}$ is very nearly the same as the corresponding expression in the two-dimensional problem for most of the smaller values of T^*/T_∞. As T^*/T_∞ becomes small and $K \to 1$ a comparison of (4.16) with (7.7) shows that $a/2A^3$ tends to the same limit $(24/5\sigma_1)^{1/3}$ in both cases.

Consequently the relation between S and P is the same as that for the two-dimensional problem to the degree of approximation used here, and the relation between S and P remains $S = 0{\cdot}24P + 10{\cdot}91(P\alpha)^{2/3}$. The results for the axisymmetric problem are similar to those for the two-dimensional case. The relation between P and S is the same so that the shielding ratio $R_2 = (1+\bar{g}_2P)^{-1}$ as a function of S is the same as that for the two-dimensional problem. The water layer and thermal layer thicknesses are less and the melting rate is greater than in the corresponding two-dimensional flow (the factor involved differs from $\sqrt{2}$ mainly because different average values of a and A were used even though $(a/A^3)^{1/3}$ has the same average values).

8. Some general considerations

The flow past a melting body has been considered for the case of a semi-infinite body of ice. With this simple geometry the full Navier–Stokes equation (with certain restrictions) has been solved in an approximate manner. However, if we first make the boundary layer approximation and interpret x, z as coordinates measured along and normal to the body we again arrive at the ordinary differential equations (4.6) and (4.8); thus the results may be applied to any body of large radius of curvature (see, for example, Levy & Seban 1953).

In most applications the high temperature T_∞ is produced by a strong shock wave and since both $\gamma - 1$ and M would be small, (M measured behind the shock) terms of $O((\gamma-1)M^2)$ would be insignificant.

Since the rate of melting depends only on the rate of transfer of heat and the components of stress at the air–liquid interface (for given conditions at $z = \pm\infty$ and given flow properties), the method may be extended to include real gas effects in the air. In this connection Hayes (1956) has shown that these effects may be taken into account approximately in such a way that the stagnation point flows have similar solutions and the boundary layer equations once again reduce to ordinary differential equations.

The more general problem of the melting of a body of finite length has no steady solution and the equations of motion reduce to partial differential equations; when the body length is large compared with its thermal boundary layer thickness the unsteady effects may be expected to be small and some account may be taken of them by a perturbation of the steady state solution.

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